On the Existence of Anonymous and Balanced Mechanisms Implementing the Lindahl Allocations

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Summary. In this note, we discuss the existence of anonymous and balanced mechanisms to implement the Lindahl allocations. We obtain an impossibility result for the class of mechanisms defining an homeomorphism between the message space and the allocation space.

1. Introduction.

The economic literature highlights a sharp contrast between the properties of economic mechanisms implementing the Lindahl allocations, in small (i.e., with two agents) and large economies (i.e., with more than two agents).

Usually, an economic mechanism is required to be balanced, individually feasible and continuous. It is said to be balanced if both equilibrium and disequilibrium outcomes induce neither surplus nor deficit of the numeraire. It is said to be individually feasible if both equilibrium and disequilibrium outcomes belong to the consumption sets of every agent. These conditions ensure that the outcome of the mechanism remain always feasible. A justification of continuity of the outcome function is that a Nash equilibrium must be robust to unwanted deviation or "trembles" in strategies (de Trenqualye, 1994).

In small economies, these conditions can never be satisfied altogether, if the mechanism fully implements in Nash equilibrium the Lindahl correspondence (i.e., any Nash equilibrium yields a Lindahl allocation and, conversely, any Lindahl allocation can be obtained as a Nash equilibrium). Kwan and Nakamura (1987) prove that a balanced mechanism implementing the Lindahl correspondence cannot be continuous. Intuitively, this is due to a basic incompatibility between Lindahl implementation, which requires that the players must not be able to affect their share in the cost of the public good, and balancedness, which implies that they actually are. However, Miura (1982) provides a discontinuous game form, which is balanced and fully implements the Lindahl correspondence.

In large economies, mechanisms exist that satisfy all the desiderata at the same time (Hurwicz, 1979; Tian, 1989; Walker, 1981). Indeed, with at least three players, the conflict just outlined can be overcome, thanks to the use of cycles in the outcome function. Hurwicz (1986, p.1468) illustrates the general principle: "One may think of agents as arranged in a circle, with each agent setting the price (acting in effect as an auctioneer) for his/her neighbors".

As a corollary, the existing mechanisms, since there are based on such cycles, fail to be anonymous. (An outcome function is said anonymous if the commodity bundles of the individuals are invariant with any permutation of their indexes.). In this note, we prove that this is not fortuitous. Precisely, we show the incompatibility between the conditions of anonymity, balancedness and Lindahl implementation, for a class of mechanisms defining an homeomorphism between the message space and the allocation space.

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The remainder of the note is organized as follows. Section 2 formalizes the economic model and gives the main definitions. Our impossibility result is proven in Section 3.

2. Notations and Definitions.

We consider an economy with one private good x, one public good y and n consumers, indexed i. We assume that the public good can be produced using the private one as an input, under a constant returns to scale technology. We normalize the units so that one unit of public good costs one unit of private good. Each consumer is characterized by his consumption set $\mathbb{R}^2_+$, his initial endowment $w_i > 0$ of the private good (none in the public good), and his preference ordering R_i, defined over $\mathbb{R}^2_+$. The set of all such economies is denoted E, with generic elements $e = (R_i, w_i)_{i=1}^{n}$.

An allocation is a vector $((x_i)_{i=1}^{n}, y) \in \mathbb{R}^{n+1}$ (giving the consumptions of the private and public goods by each consumer). For all $e \in E$, a Lindahl equilibrium is a vector of personal prices $(p_i^*)_{i=1}^{n}$, with $\Sigma_{i=1}^{n} p_i^* = 1$, and an allocation $((x_i^*)_{i=1}^{n}, y^*)$, such that:

$(x_i^*, y^*)\ R_i\ (x_i, y)$, for all $(x_i, y) \in \mathbb{R}^2_+$ such that $x_i + p_i^*\, y \leq w_i$, for all i,

$\Sigma_{i=1}^{n} x_i^* + y^* = \Sigma_{i=1}^{n} w_i$.

The allocation $((x_i^*)_{i=1}^{n}, y^*)$ is then called a Lindahl allocation (LA) of e.

For all $e \in E$, the set of Lindahl allocations is denoted $L(e)$. L is called a Lindahl correspondence.

A mechanism is a pair $(\times_{i=1}^{n} M_i, h)$, where M_i is a message space of agent i and h is an outcome function, mapping messages $m = (m_i)_{i=1}^{n}$ from $\mathcal{M} := \times_{i=1}^{n} M_i$ into allocations $h(m)$ in $\mathbb{R}^{n+1}$. More explicitly, we use the following notation:

$h(m) = ((w_i - T_i(m))_{i=1}^{n}, Y(m))$, for all $m \in \mathcal{M}$.

Below, it will be convenient to define the associated net trade correspondence:

$g(m) = ((T_i(m))_{i=1}^{n}, Y(m))$, for all $m \in \mathcal{M}$.

A Nash equilibrium (NE) of $(\mathcal{M}, h)$ is a joint strategy m^* such that, for all i ([2]):

$m^*\ R_i^*\ (m^*/m_i)$, for all $m_i \in M_i$,

where: $(m^*/m_i) = (m_1^*, \ldots, m_i, \ldots, m_n^*)$.

For all $e \in E$, the set of Nash equilibriums is denoted $\nu(e)$. The set of the corresponding allocations $h(\nu(e))$ is denoted $N(e)$. N is said to be a Nash correspondence.

Let us consider the following conditions about the mechanism.

Definition 1. $(\mathcal{M}, h)$ is said to be anonymous if:

(i) $M_1 = \ldots = M_n := M$,

(ii) $g(m_{\sigma(1)}, \ldots, m_{\sigma(n)}) = ((T_{\sigma(i)}(m))_{i=1}^{n}, Y(m))$, for all $m \in \mathcal{M}$,

where σ denotes any permutation of $\{1, \ldots, n\}$.

Definition 2. $(\mathcal{M}, h)$ is said to be balanced if:

([2]) The preferences R_i^* of i over $\mathcal{M}$ are defined by: $m'\ R_i^*\ m \Leftrightarrow h_i(m')\ R_i\ h_i(m)$, where we denote $h_i(m)$ the commodity bundle of i in the allocation $h(m)$.

$\Sigma_{i=1}^{n} T_i(m) = Y(m)$, for all $m \in \mathcal{M}$.

Definition 3. $(\mathcal{M}, h)$ is said to implement the Lindahl correspondence if:

$L(e) = N(e)$, for all $e \in E$.

3. Inexistence of anonymous mechanisms.

Many mechanisms have been proposed to implement the Lindahl allocations with Nash equilibrium, including those in Hurwicz (1979), Kim (1993), Tian (1989), de Trenqualye (1994) and Walker (1981). However, they all contradict either definition 1 (Hurwicz, 1979; Tian, 1989; Walker, 1981), definition 2 (Kim, 1993), or both (de Trenqualye, 1994). This supports the belief that the three conditions could in fact be incompatible. Proposition 1 shows the incompatibility for a class of mechanisms, such that the outcome function defines an homeomorphism between the messages space and the space of allocations.

Proposition 1. If the set of environment E is rich enough, there exists no mechanism $(\mathcal{M}, h)$ which satisfies the following conditions:

(1) It is anonymous,
(2) It is balanced,
(3) It implements the Lindahl correspondence,
(4) The mapping g defines an homeomorphism between $A = \{m \in \mathcal{M}; Y(m) > 0\}$ and $Z = \{((t_i)_{i=1}^{n}, y) \in \mathbb{R}^{n+1}; \Sigma_{i=1}^{n} t_i = y > 0\}$.

Remark 1. Among the mechanisms reviewed above, only Walker (1981) satisfies condition 4. The subscription mechanism, defined by $M_i = \mathbb{R}$, for all i, $\mathcal{M} = \mathbb{R}^n$ and $g(m) = (m_i, \Sigma_{i=1}^{n} m_i)$, for all $m \in \mathbb{R}^n$, also has this property.

Proof. Assume that the set of environment E is reach enough, so that it includes all preference profiles $R = (R_i)_{i=1}^{n}$ such that, for all i, R_i is complete, transitive, strictly increasing and convex.

Lemma 1. For all $m \in A$, $T_i(m) = T_j(m)$ if, and only if, $m_i = m_j$, for all i, j.

Proof. Without loss of generality, we consider $i = 1$ and $j = 2$ in this proof.

Consider $m = (m_1, m_2, \ldots, m_n) \in A$. Let $m' = (m_2, m_1, \ldots, m_n)$, which is derived from m, by permuting the strategies of players 1 and 2. We have:

$g(m) = (T_1(m), T_2(m), \ldots, T_n(m), Y(m))$, (by definition of g)

$g(m') = (T_2(m), T_1(m), \ldots, T_n(m), Y(m))$. (by condition 1)

If we assume that $m_1 = m_2$, then $m = m'$ and $g(m) = g(m')$. This implies that $T_1(m) = T_2(m)$.

If we assume that $T_1(m) = T_2(m)$, then $g(m) = g(m')$. Now, m and m' belong to A (for $Y(m') = Y(m) > 0$) and g is a bijection between A and Z. This implies that $m = m'$ and $m_1 = m_2$. QED

Without loss of generality, fix the initial endowment $(w_i)_{i=1}^{n}$.

Define:

$Z^* = \{((t_i)_{i=1}^{n}, y) \in \mathbb{R}^{n+1}; 0 < t_i < w_i$, for all i, and $\Sigma_{i=1}^{n} t_i = y\}$,

$A^* = \{m \in \mathcal{M}; g(m) \in Z^*\}$.

By condition 4, g defines an homeomorphism between A^* and Z^* (as $Z^* \subset Z$). Since condition 2 implies that:

$A^* = g^{-1}(\{((t_i)_{i=1}^{n}, y) \in \mathbb{R}^{n+1}; 0 < t_i < w_i$, for all $i\})$,

A^* is open, as the preimage of an open set by the continuous mapping g. Let B be a base of M, with generic elements denoted O. Let $\mathcal{B}$ be the subset of B^n, such that $\times_{i=1}^{n} O_i \subset A^*$. As A^* is open in the product topology $\mathcal{M}$, $\mathcal{B}$ is an open covering of A^*.

Lemma 2. Let $z \in Z^*$ be such that $z = (t, \ldots, t, y)$. Let $m = g^{-1}(z) \in A^*$ be the corresponding message. By lemma 1, $m = (a, \ldots, a)$, for some $a \in M$. An open (square) neighbourhood U of m exists such that $U = \times_{i=1}^{n} O \subset A^*$ and $V = h(U) \subset Z^*$ is an open neighbourhood of z.

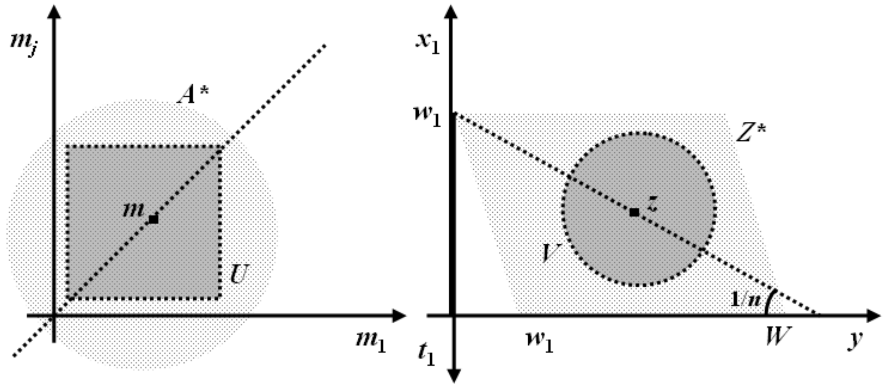
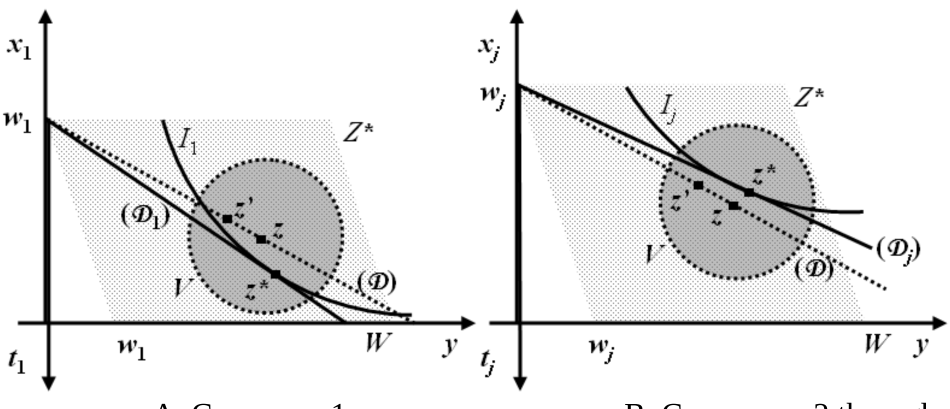


Figure 1. Illustration of Lemma 2.

Proof. Take $z = (t, \ldots, t, y) \in Z^*$. Let $m = g^{-1}(z) \in A^*$. Since $T_1(m) = \ldots = T_n(m) = t$, by lemma 1, $m = (a, \ldots, a)$, for some $a \in M$. We can find $W = (\times_{i=1}^{n} O_i) \in \mathcal{B}$, such that $m \in W \subset A^*$ (for $\mathcal{B}$ is an open covering of A^*). Choose O in $\{O_1, \ldots, O_n\}$, such that $O \subseteq O_i$, for all i. Define $U = \times_{i=1}^{n} O$. It is clear that U contains m, is open and belongs to A^*. As U is open and g^{-1} is continuous, $V = g(U)$ is open. The fact that V is a subset of Z^* containing z is immediate. QED

Lemma 3. Pick $z^* = (T^*, t^*, \ldots, t^*, y^*) \in V$, such that $T^* > t^*$. Let $m^* = g^{-1}(z^*) \in U$ be the associated message. One can find $e \in E$ such that $h(m^*) \in L(e)$, but $m^* \notin v(e)$.



A. Consumer 1. B. Consumers 2 through n.

Figure 2. Illustration of Lemma 3.

Proof. Since V is an open neighbourhood of $z = (t, \ldots, t, y)$, one can find $z^* = (T^*, t^*, \ldots, t^*, y^*) \in V$, with $T^* > t^*$. Let $m^* = g^{-1}(z^*) \in U$. Using $T^* > t^*$ together with condition 2, it is clear that:

- for player 1: $T_1(m^*) = T^* > (1/n)\, y^* = (1/n)\, Y(m^*)$,
- for the other players: $T_j(m^*) = t^* < (1/n)\, y^* = (1/n)\, Y(m^*)$. $(j = 2, \ldots, n)$

From this, z^* lies below (resp. above) line $(\mathcal{D})$ in Figure 2.A (resp. Figure 2.B), where $(\mathcal{D})$ is the line representing $t_i = (1/n)\, y$, $i = 1, \ldots, n$ (or, equivalently, $x_i + (1/n)\, y = w_i$, $i = 1, \ldots, n$).

By lemma 1, $m^* = (A^*, a^*, \ldots, a^*)$, for some $A^*, a^* \in M$. By a unilateral deviation, player 1 can reach the strategy profile $m' = (a^*, a^*, \ldots, a^*) \in U$. The corresponding net trade $z' = g(m')$ has the form $(t', \ldots, t', y')$, by lemma 1, and belongs to V. From condition 2:

- for all players: $T_i(m') = t' = (1/n)\, y' = (1/n)\, Y(m')$. $(i = 1, \ldots, n)$

Hence, z' lies somewhere on line $(\mathcal{D})$ in both Figures 2.A and 2.B.

Now, there exists $e \in E$ such that I_1, in Figure 2.A, and I_j, in Figure 2.B, are indifference curves of consumers 1 and j ($j = 2, \ldots, n$), respectively. Then, the allocation $h(m^*)$, corresponding to z^*, is a LA of e (with the Lindahl prices given by the slopes of the lines $(\mathcal{D}_1)$ and $(\mathcal{D}_j)$, in Figures 2.A and 2.B, respectively). One the other hand, since z' lies strictly above I_1 in Figure 2.A, the allocation $h(m')$, corresponding to z', is such that $(w_1 - T_1(m'), Y(m'))\ P_1\ (w_1 - T_1(m^*), Y(m^*))$ and $m'\ P_1^*\ m^*$. Hence, m^* is not a NE. QED

Now, to complete the proof of proposition 1, read z^* as the allocation $h(m^*)$ (the contradiction with the notations in Lemma 3 should not be confusing). Since A^* and Z^* are in bijection through h, m^* is the unique strategy implementing z^*. Hence, for the environment $e \in E$ considered in Lemma 3, we get $z^* \in L(e)$, but $z^* \notin N(e)$, which contradicts condition 3. **QED**

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