# On the Implementability of the Lindahl Correspondence by means of an Anonymous Mechanism

July 12, 2008

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Keywords: Lindahl equilibrium; economic mechanism.
Classification JEL : D70 ; H41.

**Summary.** This paper deals with the existence of anonymous mechanisms to realize the Lindahl correspondence. We first provide a continuous (but not smooth) and weakly balanced mechanism, which satisfies the two conditions. We then remark that it satisfies a property (see property 1'), which is in fact stronger than anonymity. Finally, we prove that if a mechanism has this property, is weakly balanced and implements the Lindahl correspondence, then it cannot be smooth.

## 1. Introduction.

The economic literature highlights a sharp contrast between the properties of economic mechanisms implementing the Lindahl allocations, in small (i.e., with two agents) and large economies (i.e., with more than two agents).

Usually, an economic mechanism is required to be (weakly) balanced, individually feasible and smooth (or continuous). It is said to be (resp. weakly) balanced if both equilibrium and disequilibrium outcomes induce neither surplus nor deficit (resp. no deficit) of the numeraire. It is said to be individually feasible if both equilibrium and disequilibrium outcomes belong to the consumption sets of every agent. These conditions ensure that the outcome of the mechanism remain feasible all the time. A justification of continuity or smoothness is that Nash equilibrium strategies must be robust to small error in prediction and/or unwanted deviation or "trembles" in strategies (de Trenqualye, 1994). Moreover, Kwan and Nakamura (1987) argue that if the outcome function is smooth, the agents can approximate prediction error linearly to determine their optimal amount of information to be collected.

In small economies, it is well known that these conditions cannot be satisfied altogether, if the mechanism also (fully) implements with Nash equilibrium the Lindahl correspondence (i.e., if any Nash equilibrium yields a Lindahl allocation and, conversely, any Lindahl allocation can be obtained as a Nash equilibrium). Precisely, Kwan and Nakamura (1987) proved that a balanced (resp. weakly balanced) mechanism implementing the Lindahl correspondence is necessarily discontinuous (resp. non smooth). Intuitively, this is due to a basic conflict between Nash implementation, which requires that the players must not be able to affect their share in the cost of the public good, and balancedness, which implies that they actually are. Moreover, the conditions in Kwan and Nakamura (1987) cannot be weakened, since Miura (1982) provided a discontinuous and balanced mechanism, and de Trenqualye (1994) constructed a continuous and weakly balanced one, both (fully) implementing the Lindahl correspondence.

In large economies, however, many mechanisms have been proposed that satisfy all the conditions (Hurwicz, 1979; Tian, 1989, 1990; de Trenqualye, 1994; Walker, 1981). They all overcome the incompatibility just outlined, thanks to the use of cycles in the outcome

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function. The general principle is illustrated by Hurwicz (1986, p.1468) as follows: “One may think of agents as arranged in a circle, with each agent setting the price (acting in effect as an auctioneer) for his/her neighbors”. In large economies, where cycles involving more than two players can be defined, this trick is sufficient to eliminate the contradiction outlined above between Nash implementation and balancedness.

As a corollary, under these mechanisms, the consumption bundle of a player will depend on the strategies played by his immediate neighbours in the cycle, and hence will not be invariant by permutation of the individuals’ indexes. That is, these mechanisms fail to be anonymous. (Roughly speaking, a mechanism is said to be anonymous if any permutation of the individuals’ indexes leaves unchanged their commodity bundles, both in and out of equilibrium.)

Now, it seems intuitive that the condition of anonymity, as it constrains the outcome function to have a similar shape as the ones available for the two-agents economies, is likely to bring back the trade-off between Nash implementation and balancedness. This casts doubt on the existence of mechanisms which, in large economies, satisfy the conditions discussed previously, while being anonymous at the same time.

In this paper, we give two results in relation with this existence problem.

We first construct an anonymous, continuous (but not smooth) and weakly balanced mechanism, which (fully) implements the Lindahl correspondence. We emphasize in passing the following nice property. Under this mechanism, a strategy of a player is two-dimensional and (we argue) can be read as an announcement of a marginal willingness to pay/personal price and a demand for the public good. Indeed, in equilibrium, each agent really obtains the commodity bundle corresponding to his report (This property is referred to as forthrightness in Saijo and al., 1996) and reveals honestly his willingness to pay and his demand for the public good (see remark 1).

We then remark that this mechanism satisfies a property, which is in fact stronger than just anonymity (see property 1’). Finally, we prove that if a mechanism satisfies this stronger condition, is weakly balanced and (fully) implements the Lindahl correspondence, then it cannot be smooth.

The remainder of the paper is organized as follows. Section 2 formalizes the economic model and gives the main definitions. In section 3, we define the new mechanism and derive its properties. In Section 4, we prove the impossibility result. Section 5 concludes the paper.

## 2. Notations and Definitions.

We consider an economy with one private good $x$, one public good $y$ and $n$ consumers, indexed $i$. We assume that the public good can be produced using the private one as an input, under a constant returns to scale technology. We normalize the units so that one unit of public good costs one unit of private good. Each consumer is characterized by his consumption set $X_i$, is initial endowment $w_i > 0$ of the private good (none in the public good), and his preference ordering $R_i$, defined over $X_i$ ([2]). The set of admissible economies is denoted $E$, with generic elements $e = (R_i, w_i)_{i=1}^{n}$.

An allocation is a vector $((x_i)_{i=1}^{n}, y) \in \mathrm{IR}^{n+1}$, giving the consumptions of the private and public goods by the consumers. For all $e \in E$, a Lindahl equilibrium (LE) is a vector of personal prices $(p_i^*)_{i=1}^{n}$, with $\Sigma_{i=1}^{n} p_i^* = 1$, and an allocation $((x_i^*)_{i=1}^{n}, y^*)$, such that:

$(x_i^*, y^*) \; R_i \; (x_i, y)$, for all $(x_i, y) \in X_i$ such that $x_i + p_i^* y \le w_i$, for all $i$,

$\Sigma_{i=1}^{n} x_i^* + y^* = \Sigma_{i=1}^{n} w_i$.

The allocation $((x_i^*)_{i=1}^{n}, y^*)$ is then called a Lindahl allocation (LA).

([2]) In section 3, we disregard the question of individual feasibility, letting $X_i = \mathrm{IR}^2$. See remark 2.

For all $e \in E$, the set of Lindahl allocations is denoted $L(e)$. $L$ is said to be a Lindahl correspondence.

A mechanism is a pair $(M, h)$, consisting of a message space $M = \times_{i=1}^{n} M_i$, where $M_i$ is player $i$'s message space, and an outcome function $h$. The outcome function $h$ associates to any message $m \in M$, an allocation $h(m) \in \mathbb{R}^{n+1}$. More explicitly, we use the following notation:

$$h(m) = ((X_i(m))_{i=1}^{n}, Y(m)), \text{ for all } m \in M.$$

Below, it will be convenient to define, for all $m \in M$:

$$T_i(m) = w_i - X_i(m), \text{ for all } i,$$
$$g(m) = ((T_i(m))_{i=1}^{n}, Y(m)).$$

The pair $(M, h)$ defines a game form, where the players are the consumers $i = 1, \dots, n$, the strategic set of $i$ is $M_i$, and his preference over $M$, denoted $R_i^*$, follows from his preference $R_i$ over $X_i$, and from the outcome function $h$ ([3]). A strategy of a player $i$ will be denoted $m_i$. A Nash equilibrium (NE) of $(M, h)$ is a joint strategy $m^*$ such that, for all $i$:

$$m^* \, R_i^* \, (m^*/m_i), \text{ for all } m_i \in M_i,$$

where: $(m^*/m_i) = (m_1^*, \dots, m_i, \dots, m_n^*)$.

For all $e \in E$, the set of Nash equilibriums is denoted $v(e)$. The set of the corresponding allocations $h(v(e))$ is denoted $N(e)$. $N$ is said to be a Nash correspondence.

Let us consider the following conditions about the mechanism.

**Definition 1.** A mechanism $(M, h)$ is said to be anonymous if:

(i) $M_1 = \dots = M_n$,

(ii) $g(m_{\sigma(1)}, \dots, m_{\sigma(n)}) = ((T_{\sigma(i)}(m))_{i=1}^{n}, Y(m))$, for all $m \in M$,

where $\sigma$ denotes any permutation of the set $\{1, \dots, n\}$.

**Definition 2.** A mechanism $(M, h)$ is said to be weakly balanced if:

$$\Sigma_{i=1}^{n} X_i(m) + Y(m) \leq \Sigma_{i=1}^{n} w_i, \text{ for all } m \in M.$$

(It is said balanced if the inequality $\leq$ is replaced by the equality $=$.)

**Definition 3.** A mechanism $(M, h)$ is said to (fully) implement the Lindahl correspondence if:

$$L(e) = N(e), \text{ for all } e \in E.$$

## 3. Continuous implementation.

In this section, we define a new mechanism to (fully) implement the Lindahl correspondence. We show that it is anonymous, continuous and weakly balanced. We then state an interesting property of this game form, dealing with its interpretation.

**Definition 4.** For $n \geq 2$, consider the mechanism $(M, h)$ such that:

- the message space is $M = \mathbb{R}^{2n}$;
- the strategic space of player $i$ is $M_i = \mathbb{R}^2$, with elements denoted $m_i = (p_i, y_i)$;
- for all $m \in M$, the outcome function is defined by:

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([3]) Formally, for any two messages $m$ et $m'$, the preferences of player $i$ over $M$ are defined by: $m' \, R_i^* \, m \Leftrightarrow (X_i(m'), Y(m')) \, R_i \, (X_i(m), Y(m))$.

$X_i(m) = w_i - p_i(m)\, Y(m) - (1 - 1/n)\, |f_i(m)|\, (|Y(m)| + \varepsilon)$, with $\varepsilon > 0$, for all $i$,

$Y(m) = \Sigma_{i=1}^{n} y_i/n$,

where, for all $m$:

$p_i(m) = 1 - \Sigma_{j\neq i}\, p_j$, for all $i$,

$f_i(m) = \Sigma_{i=1}^{n} p_i - 1 + \Sigma_{j\neq i}\, y_j/(n-1) - y_i$.

Under this mechanism, for all $i$, the strategies $p_i$ et $y_i$ should be interpreted as player $i$'s report, respectively, of his marginal willingness to pay and his demand for the public good (see remark 1 for a justification). The players contribute to the cost of provision of the public good, by paying a personal price $p_i(m)$, on each unit, and, possibly, a penalty $(1 - 1/n)\, |f_i(m)|\, (|Y(m)| + \varepsilon)$. The personal price $p_i(m)$ of player $i$ is the difference between the marginal cost of production of the public good and the sum of the marginal propensities to pay announced by the other players, i.e. $1 - \Sigma_{j\neq i}\, p_j$. The supply of public good $Y(m)$ is set equal to the average demand, i.e. $(1/n)\, \Sigma_{i=1}^{n} y_i$.

**Proposition 1**. Let $E^*$ denote the set of economies $e = (R_i, w_i)_{i=1}^{n}$ such that, for each $i$, $R_i$ is complete, transitive and strictly increasing in the private good. Assume that $E \subseteq E^*$. The mechanism $(M, h)$ given in definition 4, is anonymous, continuous, weakly balanced and (fully) implements the Lindahl correspondence.

**Proof.**

Consider the mechanism $(M, h)$ defined in definition 4.

It is clear that $(M, h)$ is anonymous and continuous.

To see that $(M, h)$ is weakly balanced, note that:

$\Sigma_{i=1}^{n} (1 - 1/n)\, |f_i(m)|\, (|Y(m)| + \varepsilon) \geq (n-1)\, (\Sigma_{i=1}^{n} p_i - 1)\, Y(m)$,

from which one can prove that $\Sigma_{i=1}^{n} X_i(m) + Y(m) \leq \Sigma_{i=1}^{n} w_i$.

**Lemma 1.** If $m^*$ is a NE, then $h(m^*)$ is a Lindahl allocation.

**Proof.** Consider an economy $e \in E$ and a NE $m^* = (p_i^*, y_i^*)_{i=1}^{n}$ of $(M, h)$.

Assume, by way of contradiction, that $i$ exists such that $f_i(m^*) \neq 0$.

Letting $m_i = (1 - \Sigma_{j\neq i}\, p_j^* - \Sigma_{j\neq i}\, y_j^*/(n-1) + y_i^*, y_i^*)$, we get:

$p_i(m^*/m_i) = p_i(m^*) = 1 - \Sigma_{j\neq i}\, p_j^*$,

$|f_i(m^*/m_i)| = 0 < |f_i(m^*)|$,

$Y(m^*/m_i) = Y(m^*)$.

Since $R_i$ is strictly increasing in the private good, it follows that:

$$(w_i - p_i(m^*/m_i)\, Y(m^*/m_i) - (1 - 1/n)\, |f_i(m^*/m_i)|\, (|Y(m^*/m_i)| + \varepsilon), Y(m^*/m_i))$$
$$P_i\, (w_i - p_i(m^*)\, Y(m^*) - (1 - 1/n)\, |f_i(m^*)|\, (|Y(m^*)| + \varepsilon), Y(m^*)).$$

Therefore, $(m^*/m_i)\, P_i^*\, m^*$ and $m^*$ is not a NE.

By contradiction, we have thus proven that:

$$f_i(m^*) = \Sigma_{i=1}^{n} p_i^* - 1 + \Sigma_{j\neq i}\, y_j^*/(n-1) - y_i^* = 0, \text{ for all } i. \quad (1)$$

Now, notice that $\Sigma_{i=1}^{n} f_i(m) = n\, (\Sigma_{i=1}^{n} p_i - 1)$, for all $m$. From (1), it follows that:

$$\Sigma_{i=1}^{n} p_i^* = 1. \quad (2)$$

Substituting in each $f_i(m^*)$, we get:

$$f_i(m^*) = \Sigma_{j\neq i}\, y_j^* - (n-1)\, y_i^* = 0, \text{ for all } i. \quad (3)$$

Finally, (1), (2) and (3) implies that, for all $i$:

$$p_i^* = 1 - \Sigma_{j\neq i}\, p_j^* = p_i(m^*), \quad (4)$$

$y_i^* = (1/n)\ \Sigma_{i=1}^{n}\ y_i^* = Y(m^*).$ (5)

Now, let $((x_i^*)_{i=1}^{n}, y^*) = h(m^*)$. Since $f_i(m^*) = 0$, for all $i$, we have:

$$((x_i^*)_{i=1}^{n}, y^*) = ((w_i - p_i(m^*)\ Y(m^*))_{i=1}^{n}, Y(m^*)).$$

This allocation is balanced, for $\Sigma_{i=1}^{n}\ p_i(m^*) = \Sigma_{i=1}^{n}\ p_i^* = 1$ implies that:

$$\Sigma_{i=1}^{n}\ x_i^* + y^* = \Sigma_{i=1}^{n}\ (w_i - p_i(m^*)\ y^*) + y^* = \Sigma_{i=1}^{n}\ w_i + (1 - \Sigma_{i=1}^{n}\ p_i(m^*))\ y^* = \Sigma_{i=1}^{n}\ w_i.$$

Since the players are never satiated in the private good, without loss of generality, assume (by way of contradiction) that $i$ and $(x_i, y)$ exist such that:

$$(x_i, y)\ P_i\ (x_i^*, y^*) \text{ and } x_i + p_i(m^*)\ y = w_i.$$

Substituting (using $x_i = w_i - p_i(m^*)\ y$ and $(x_i^*, y^*) = (w_i - p_i(m^*)\ Y(m^*), Y(m^*))$), we find:

$$(w_i - p_i(m^*)\ y, y)\ P_i\ (w_i - p_i(m^*)\ Y(m^*), Y(m^*)).$$

Letting $m_i = (1 - \Sigma_{j\neq i}\ p_j^* + n\ (y - \Sigma_{j\neq i}\ y_j^*/(n - 1)), n\ y - \Sigma_{j\neq i}\ y_j^*)$, we get:

$$p_i(m^*/m_i) = p_i(m^*) = 1 - \Sigma_{j\neq i}\ p_j^*,$$
$$|f_i(m^*/m_i)| = |f_i(m^*)| = 0,$$
$$Y(m^*/m_i) = y.$$

Substituting, we obtain:

$$(w_i - p_i(m^*/m_i)\ Y(m^*/m_i) - (1 - 1/n)\ |f_i(m^*/m_i)|\ (|Y(m^*/m_i)| + \varepsilon), Y(m^*/m_i))$$
$$P_i\ (w_i - p_i(m^*)\ Y(m^*) - (1 - 1/n)\ |f_i(m^*)|\ (|Y(m^*)| + \varepsilon), Y(m^*)).$$

Therefore, $(m^*/m_i)\ P_i^*\ m^*$ and $m^*$ is not a NE.

Finally, we proved that $((x_i^*)_{i=1}^{n}, y^*)$ satisfies:

$$(x_i^*, y^*)\ R_i\ (x_i, y), \text{ for all } (x_i, y) \text{ such that } x_i + p_i(m^*)\ y \leq w_i, \text{ for all } i,$$
$$\Sigma_{i=1}^{n}\ x_i^* + y^* = \Sigma_{i=1}^{n}\ w_i.$$

Hence, it is a Lindahl allocation. QED

**Lemma 2.** If $(p_i^*)_{i=1}^{n}$ and $((x_i^*)_{i=1}^{n}, y^*)$ is a LE, then $m^* = (p_i^*, y^*)$ is the unique NE such that $h(m^*) = ((x_i^*)_{i=1}^{n}, y^*)$.

**Proof.** Consider an economy $e \in E$ and a LE $(p_i^*)_{i=1}^{n}$ (with $\Sigma_{i=1}^{n}\ p_i^* = 1$) and $((x_i^*)_{i=1}^{n}, y^*)$ of $e$. By definition:

$(x_i^*, y^*)\ R_i\ (x_i, y)$ for all $(x_i, y)$ such that $x_i + p_i^*\ y \leq w_i$, for all $i$, (6)

$$\Sigma_{i=1}^{n}\ x_i^* + y^* = \Sigma_{i=1}^{n}\ w_i.$$

Because the players are never satiated with the private good, (6) implies, for all $i$ and all $y$:

$(w_i - p_i^*\ y^*, y^*)\ R_i\ (w_i - p_i^*\ y, y).$ (7)

Letting $m^* = (p_i^*, y^*)_{i=1}^{n}$, we obtain ([4]):

$f_i(m^*) = 0$, for all $i$, (for $\Sigma_{i=1}^{n}\ p_i^* = 1$ and $y_i^* = \Sigma_{j\neq i}\ y_j^*/(n - 1)$),

$p_i(m^*) = 1 - \Sigma_{j\neq i}\ p_j^* = p_i^*$, for all $i$, (for $\Sigma_{i=1}^{n}\ p_i^* = 1$),

$Y(m^*) = y^*.$

Substituting in (7), we get, for all $i$ and all $y$:

$(w_i - p_i(m^*)\ Y(m^*) - (1 - 1/n)\ |f_i(m^*)|\ (|Y(m^*)| + \varepsilon), Y(m^*))\ R_i\ (w_i - p_i(m^*)\ y, y).$ (8)

Now, for all $i$, let $m_i = (p_i, n\ y - \Sigma_{j\neq i}\ y_j^*)$, where $p_i$ and $y$ are any real numbers. By definition:

$$p_i(m^*/m_i) = p_i(m^*) = 1 - \Sigma_{j\neq i}\ p_j^*,$$
$$|f_i\ (m^*/m_i)| \geq |f_i(m^*)| = 0,$$

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([4]) The strategy profile $m^* = (p_i^*, y^*)_{i=1}^{n}$ is the unique solution to $p_i(m) = p_i^*$, for all $i$, and $Y(m) = y^*$, such that $f_i(m) = 0$, for all $i$. From lemma 1, we know that $f_i(m) = 0$, for all $i$, is necessary at a NE. Uniqueness follows.

$Y^*(m^*/m_i) = y$.

Substitute $p_i(m^*/m_i) = p_i(m^*)$ and $Y^*(m^*/m_i) = y$ into the RHS of (8). Then, for all $i$ and $m_i$:

$$(w_i - p_i(m^*)\, Y(m^*) - (1 - 1/n)\, |f_i(m^*)|\, (|Y(m^*)| + \varepsilon),\, Y(m^*))$$
$$R_i\, (w_i - p_i(m^*/m_i)\, Y(m^*/m_i),\, Y(m^*/m_i)).$$

As $R_i$ is strictly increasing in the private good and $(1 - 1/n)\, |f_i(m^*/m_i)\,|\, (|Y(m^*/m_i)| + \varepsilon) \geq 0$, we have, for all $i$ and all $m_i$:

$$(w_i - p_i(m^*/m_i)\, Y(m^*/m_i),\, Y(m^*/m_i))$$
$$R_i\, (w_i - p_i(m^*/m_i)\, Y(m^*/m_i) - (1 - 1/n)\, |f_i(m^*/m_i)|\, (|Y(m^*/m_i)| + \varepsilon),\, Y(m^*/m_i)).$$

By transitivity of $R_i$:

$$(w_i - p_i(m^*)\, Y(m^*) - (1 - 1/n)\, |f_i(m^*)|\, (|Y(m^*)| + \varepsilon),\, Y(m^*))$$
$$R_i\, (w_i - p_i(m^*/m_i)\, Y(m^*/m_i) - (1 - 1/n)\, |f_i(m^*/m_i)|\, (|Y(m^*/m_i)| + \varepsilon),\, Y(m^*/m_i)).$$

Therefore, for all $i$:

$$m^*\, R_i^*\, (m^*/m_i), \text{ for all } m_i \in M_i,$$

which means that $m^*$ is a NE of $(M, h)$. QED

**Remark 1.** Recall that we interpreted a message $m_i = (p_i, y_i)$ of $i$ as his report of his marginal willingness to pay $p_i$ and his demand for the public good $y_i$. A very interesting feature of the mechanism defined above is that this reading of the individual messages fits perfectly the properties of the Nash equilibriums. Indeed, let $m^* = (p_i^*, y_i^*)_{i=1}^n$ be a Nash equilibrium of some economy $e \in E$. In the proof of lemma 1, it is shown that:

(i) For all $i$, $f_i(m^*) = 0$, $p_i(m^*) = p_i^*$ and $Y(m^*) = y_i^*$ (see equations (1), (4) and (5));

(ii) $(p_i(m^*))_{i=1}^n$ and $h(m^*)$ defines a Lindahl equilibrium of $e$.

From property (i), at a Nash equilibrium, the agents capture perfectly their situation if they read their individual strategies as suggested. Indeed, by definition of $h$, a player $i$ gets the commodity bundle $(w_i - p_i^*\, y_i^*, y_i^*)$. Therefore, according to our interpretation, he is supplied the amount of public good he demanded at a personal price equal to the marginal willingness to pay he reported. This property is reminiscent to the condition of forthrightness introduced by Saijo and al. (1996) ([5]). Until now, we only proved that reading the individual strategies as proposed provides the participants with an appealing way to compute any Nash equilibrium allocation. However, property (ii) shows that our interpretation is also true and informative (to either the participants or an outsider). Indeed, as $(p_i(m^*))_{i=1}^n$ and $h(m^*)$ is a Lindahl equilibrium of $e$, $p_i(m^*)$ and $Y(m^*)$ measure agent $i$'s marginal willingness to pay and demand for the public good, respectively. Now, from (i), we have $m_i^* = (p_i^*, y_i^*) = (p_i(m^*), Y(m^*))$. Hence, in a Nash equilibrium, the players directly and truthfully reports their marginal willingness to pay/personal price and demand for the public good through their strategies.

**Remark 2.** The assumption $X_i = \mathbb{R}^2$, for all $i$, is necessary for the game form given in definition 4 to be well-defined. For example, if $X_i = \mathbb{R}^2_+$, there exists some strategic profiles $m$ leaving $i$ outside his consumption set $X_i$. Many mechanisms share this defect, including those by Hurwicz (1979), Walker (1981), Kim (1993) and de Trenqualye (1994) ([6]). Hurwicz and al. (1984) overcome this weakness, but their game form makes use of a huge and complex

([5]) Saijo and al. (1996) analysed the class of mechanisms such that one component of the message space of each participant is interpretable as his consumption bundle. Then, forthrightness requires that in equilibrium, each agent receives what he has announced as his own consumption bundle.

([6]) Hurwicz (1979) obtained a well-defined game form by constructing an extended preference ordering for each player, such that any $(x_i, y) \in X_i$ is superior to all $(x_i, y) \notin X_i$.

message space (Hurwicz, 1986). To the best of our knowledge, only Tian (1989, 1990) supplies a simple mechanism which avoids this problem ([7]).

**4. Impossibility of a smooth implementation.**

The literature provides (weakly) balanced and smooth mechanisms, which implement the Lindahl correspondence (Hurwicz, 1979; Walker, 1981). However, none is anonymous. On the other hand, in Section 3, we defined an anonymous and weakly balanced mechanism, which implements the Lindahl correspondence. However, it is not smooth. This observation supports the belief that a trade-off might exist between anonymity and smoothness. In this section, we obtain an impossibility result, which is a first step toward a confirmation of this conjecture. Our proof closely follows Kwan and Nakamura (1987).

We have seen above that the mechanism $(M, h)$, given in definition 4, is anonymous. In fact, it is more closely characterized by the following property, which is actually stronger.

**Definition/Property 1'.** The mechanism $(M, h)$ given in definition 4 satisfies the following conditions:

$M_1 = \ldots = M_n$,

and, for all $m \in M$,

$T_i(m) = T(m_i, \Sigma_{j \neq i}\, m_j)$, for all $i$,

$Y(m) = G(m_i + \Sigma_{j \neq i}\, m_j)$,

where $T$ and $G$ are functions defined over $M$ and taking values in $\mathbb{R}$.

The proposition below proves a contradiction between anonymity and smoothness, provided that definition 1' is used in place of definition 1 to characterize the anonymity.

**Proposition 2.** Let $E_C$ be the set of economies such that, for all $(a_i)_{i=1}^{n} \in (0, 1)^n$, each preference $R_i$ is represented by the utility function:

$U^i(x_i, y) = x_i^{a_i}\, y^{1-a_i}$, for all $(x_i, y) \in X_i$,

and the initial endowment is fixed at a given $(w_i)_{i=1}^{n}$. Assume that $E_C \subseteq E$. Then, there exists no smooth mechanism $(M, h)$ which satisfies the definitions 1', 2 and 3.

**Proof.** For all $e \in E_C$, characterized by $a = (a_i)_{i=1}^{n} \in (0, 1)^n$ and $(w_i)_{i=1}^{n}$, $((x_i)_{i=1}^{n}, y) \in L(e)$ if and only if:

$x_i = a_i\, w_i$, for all $i$,

$y = (1 - a_i)\, w_i/p_i$,

where the personal prices $(p_i)_{i=1}^{n}$ are given by:

$p_i = ((1 - a_i)\, w_i)/(\Sigma_{i=1}^{n}\, (1 - a_i)\, w_i)$, for all $i$.

Hence, we have:

$L(E_C) = \{((x_i)_{i=1}^{n}, y) \in \mathbb{R}^{n+1};\ 0 < x_i < w_i, \text{ for all } i, \text{ and } \Sigma_{i=1}^{n}\, x_i + y = \Sigma_{i=1}^{n}\, w_i\}$.

Now, assume (by way of contradiction) that $(M, h)$ satisfies definitions 1', 2 and 3, with $h$ a smooth function.

([7]) In a paper in progress, following Tian (1990), the mechanism described in definition 4 is amended to ensure individually feasibility, both in and out of equilibrium. The (full) implementation of the Lindahl correspondence holds if the preferences are assumed strictly monotone, convex and such that any interior allocation is strictly preferred to a boundary allocation.

Let $B^0 = \mathbb{R}^{n+1}_{++}$ and $V = h^{-1}(B^0)$ ([8]). Since $B^0$ is open and $h$ is continuous, $V$ is open.

Let $v(E_C) = \{m \in M; \exists\, e \in E_C, m \in v(e)\}$. By condition 3, if $m \in v(E_C)$, then $h(m) \in L(E_C)$. Since $L(E_C) \subset B^0$, it follows that $v(E_C) \subset V$.

For all $m \in V$ (where $Y(m) > 0$), define:

$P_i(m) = (w_i - X_i(m))/Y(m)$, for all $i$.

**Lemma 3.** For all $m \in v(E_C)$, $D_i P_i(m) = 0$, for each $i$, where $D_i$ denotes the partial derivatives with respect to $m_i$.

**Proof.** Let $m \in v(E_C)$. Consider the environment $e \in E_C$ for which $m$ is a Nash equilibrium. Since $m_i$ maximizes $U^i(X_i(m), Y(m))$ over $M_i$, we have, for all $i$ ([9]):

$$D_{x_i} U^i(X_i(m), Y(m))\, D_i X_i(m) + D_y U^i(X_i(m), Y(m))\, D_i Y(m) = 0, \quad (9)$$

where $D_{x_i}$ and $D_y$ denote the partial derivatives with respect to $x_i$ and $y$, respectively.

By definition of $P_i$:

$X_i(m) + P_i(m)\, Y(m) = w_i$, for all $m \in V$.

By differentiating with respect to $m_i$, we can get:

$$D_i X_i(m) + P_i(m)\, D_i Y(m) + D_i P_i(m)\, Y(m) = 0. \quad (10)$$

Since $h(m)$ is a Lindahl allocation, we also have ([10]):

$$D_y U^i(X_i(m), Y(m))/D_{x_i} U^i(X_i(m), Y(m)) = P_i(m). \quad (11)$$

From (9), (10) and (11), we can get $D_i P_i(m)\, Y(m) = 0$. Since $h(m) \in L(E_C)$, $Y(m) > 0$ and $D_i P_i(m) = 0$. Q.E.D.

**Lemma 4.** For all $m \in v(E_C)$, $D_j P_i(m) = 0$ for each $i$ and $j$.

**Proof.** If $i = j$, it is proven in Lemma 3.

Since the mechanism is weakly balanced, we have:

$\Sigma_{j=1}^{n} P_j(m) = \Sigma_{j=1}^{n} (w_j - X_j(m))/Y(m) \geq 1$, for all $m \in V$.

Pick $m \in v(E_C)$. By Lindahl implementation, $h(m) \in L(E_C)$ and:

$\Sigma_{j=1}^{n} P_j(m) = \Sigma_{j=1}^{n} (w_j - X_j(m))/Y(m) = 1$.

From this, $m$ minimizes $\Sigma_{j=1}^{n} P_j(m)$ over the open set $V$, and satisfies the first order conditions:

$$\Sigma_{j=1}^{n} D_i P_j(m) = 0, \text{ for all } i. \quad (12)$$

From (12) and Lemma 3, it follows that:

$$\Sigma_{j \neq i} D_i P_j(m) = 0, \text{ for all } i. \quad (13)$$

Since $(M, h)$ satisfies condition 1', for all $i$, we have:

$D_1 P_i(m) = \ldots = D_{i-1} P_i(m) = D_{i+1} P_i(m) = \ldots = D_n P_i(m)$, for all $m \in V$.

Substitute $A_i(m) \equiv D_j P_i(m)$, for all $j \neq i$, in (13) to get the system:

$$\begin{aligned} 0 + A_2(m) + \ldots + A_n(m) &= 0,\\ A_1(m) + 0 + \ldots + A_n(m) &= 0,\\ &\ldots\\ A_1(m) + A_2(m) + \ldots + 0 &= 0. \end{aligned}$$

This system has a unique solution: $A_1(m) = A_2(m) = \ldots = A_n(m) = 0$. Q.E.D.

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([8]) Note that if $V$ is empty, then Lindahl implementation is trivially impossible.

([9]) Since $m \in v(E_C) \subset V$ and $V$ is open, $m$ is interior and (9) is satisfied.

([10]) Since $h(m)$ is a LA, there exists personal prices $(p_i)_{i=1}^{n}$ such that: $D_y U^i(X_i(m), Y(m))/D_{x_i} U^i(X_i(m), Y(m)) = p_i$ and $X_i(m) + p_i\, Y(m) = w_i$, for all $i$. As $h(m) \in L(E_C)$, $Y(m) > 0$ and we get $p_i = (w_i - X_i(m))/ Y(m) = P_i(m)$, for all $i$.

From Lemma 4, for all $m \in v(E_C)$, $m$ is a critical point of $P_i(m)$ ($i$ = 1, …, $n$). By Sard's theorem, $P_i(v(E_C))$ is of measure zero. But since ($M$, $h$) implements the Lindahl correspondence, $P_i(v(E_C))$ = (0,1), which is a contradiction. **Q.E.D.**

**Remark 3.** It should be noted from the proof that Proposition 2 remains true if we only require that any Nash equilibrium implements a Lindahl allocation: $N(e) \subset L(e)$, for all $e \in E$.

## 5. Conclusion.

In this paper, we have constructed an anonymous, continuous and weakly balanced mechanism to realize the Lindahl allocations. We have stated some interesting properties of it. In equilibrium, each player reveals directly and truthfully his personal price and his demand for the public good, and obtains the associated commodity bundles. This means that in equilibrium, there exists one straightforward way to interpret and process the messages used in the mechanism. One defect of our game form is that it is not smooth. However, we conjecture that this is not avoidable. Indeed, using a stronger than necessary (but nevertheless natural) definition of anonymity, we have shown that no anonymous and weakly balanced mechanism can implement the Lindahl allocations, if it is smooth. The agenda for future researches should be to prove this impossibility under a standard definition of anonymity and to check the existence (or not) of an anonymous, balanced and continuous mechanism to implement the Lindahl correspondence.

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