A Mechanism implementing the Lindahl allocations with an Interpretable Message Space

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Summary. We present a mechanism which implements the Lindahl allocations with Nash equilibrium. The players send 2-dimensional messages. A player's message can be interpreted as his report of his marginal propensity to pay and his demand for the public good. At a Nash equilibrium, the outcome of the mechanism is a Lindahl allocation and each players' equilibrium strategy is to reveal his personal price and his demand for the public good. In a quasi-linear economy, using the gradient process to formalize the behaviour of the players out-of-equilibrium, the unique stationary point of this dynamic is globally stable and coincide with the Nash equilibrium of the game form.

1. Introduction.

Implementation theory deals with the problem of designing mechanisms (i.e., pairs consisting of a message space and an outcome function) to induce individuals to reveal their private information, in order to realize some given social objectives (Hurwicz, 1986). Primarily, these social objectives refer to conditions to be satisfied by outcomes (such as the implementation of Pareto, Walras, Lindahl or no-envy allocations). However, the formal characteristics of the mechanism itself are also of importance and complementary conditions may be required (such as the equilibrium concept used in the game, the dimension of the message space, the properties of disequilibrium outcomes, the stability of equilibriums, etc.).

In economies with public goods, Lindahl allocations have attractive properties (Hurwicz, 1986). First, the existence of Lindahl equilibriums can be proved under standard conditions (Foley, 1970). Second, under the same assumptions, Lindahl allocations belong to the core of the economy (Foley, 1970). Thus, they are both Pareto-optimal and individually rational. This explains why, assuming that the consumers have perfect information, many mechanisms have been proposed to implement the Lindahl allocations, including those in Hurwicz (1979), Kim (1993), Tian (1989), de Trenqualye (1994) and Walker (1981) ([2]).

These mechanisms are in a sense equivalent, for they all realize the same social objective. However, formal differences arise with respect to: the dimension of the message space (i.e., the number of components of a message); the number of players necessary to use the mechanism; the property of anonymity (a mechanism is anonymous when outcomes are independent of the consumers' indices); the property of balance (a mechanism is balanced if both equilibrium *and* disequilibrium outcomes induce neither surplus nor deficit of the private good; Hurwicz, 1986); the individual feasibility (a mechanism is individually feasible if both equilibrium *and* disequilibrium outcomes are in the consumption set of all the consumers); the stability of the Nash equilibriums.

Walker (1981) constructs a mechanism with a message space of minimum dimension (Hurwicz, 1986), while the others use larger message spaces. Kim (1993) and de Trenqualye (1994) provide mechanisms working with two players or more, whereas Hurwicz (1979), Tian

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([2]) Exceptions are Groves and Ledyard (1977) and Falkinger (1996), whose mechanisms implement Pareto-optimal allocations.

(1989) and Walker (1981) require a minimum of three players. Only Kim (1993) proposes an anonymous mechanism. The outcome functions constructed by Hurwicz (1979), Tian (1989) and Walker (1981) are balanced, not those of Kim (1993) and de Trenqualye (1994) ([3]). Only the mechanism of Tian (1989) is individually feasible. With quasi-linear utility functions, the Nash equilibriums of the mechanisms of Kim (1993) and de Trenqualye (1994) are stable (respectively, under the 'gradient' and 'best-replay' processes). On the contrary, those of Hurwicz (1979) and Walker (1981) are unstable for any decentralized adjustment process (Kim, 1993).

The problem with these mechanisms is that they rely on abstract message spaces. That is, the messages they use cannot be related with standard economic characteristics and thus do not have an economic meaning. The aim of the present paper is therefore to provide a new mechanism whose messages can receive an obvious economic interpretation. In fact, under our mechanism, the players manipulate 2-dimensional messages, which in Nash equilibrium are directly related with their marginal willingness to pay and their demands for the public good.

To explain why we believe this feature is important, let us turn our attention to the closely related literature on planning (see, for example, Malinvaud, 1971; Drèze and de la Vallée-Poussin, 1971). Formally, the benchmark model is the same as the one used in implementation theory. That is, agents (including an auctioneer) communicate by means of messages and an outcome function associates messages with allocations. However, in the literature on planning, the agents are modelled as basic information processors, converting non-strategically data of one type into another (for example, they observe prices and announce demands). In this setting, it is immediate that the message space need to be explicitly linked with economic characteristics (marginal rates of substitution, demands, etc.) and variables (prices, quantities, etc.), so that the individuals can foresee their role during the planning process. In implementation theory, on the other hand, the players are assumed perfectly rational. In other words, by assumption, no matter the message space, they will eventually (learn how to) use it optimally to maximize their preferences. Thus, from the perspective of mechanism design, any abstract set is theoretically admissible as a message space.

In reality, if agents have low skills or if both types of agents coexist, there is a need for an intermediate approach. On the one hand, it is desirable that if necessary, the designer should be able to advise the (low-skilled) agents about one obvious way to interpret and process the messages used by the mechanism. On the other hand, the recommended behaviour must not be vulnerable to the free-riding problem. These two conditions have not yet been satisfied altogether in the literature. Indeed, in the mechanisms of Hurwicz (1979), Kim (1993), Tian (1989), de Trenqualye (1994) and Walker (1981), no appealing interpretation of *all* components of a message vector can be derived. (In fact, the Nash equilibrium messages are linear combinations of the personal prices and the level of public good, associated with the underlying Lindahl equilibrium.) Furthermore, it is well known that the literature on planning have been criticized on the ground of strategic manipulability (Lindahl, 1919; Malinvaud, 1971; Drèze and de la Vallée-Poussin, 1971).

In this paper, we present an anonymous mechanism for economies with two consumers or more, implementing with Nash equilibrium the Lindahl correspondence. Its interest is that the Nash equilibrium messages are directly related with economic characteristics and thus, the

([3]) By theorem, no smooth and balanced mechanism can fully implement with Nash equilibrium the Lindahl allocations in economies with two agents (Hurwicz, 1986). Therefore, the mechanisms in Kim (1993) and de Trenqualye (1994) being smooth cannot be balanced. Notice however that the mechanism constructed by de Trenqualye (1994) is globally (or weakly) feasible (i.e., no deficit in the private good occurs out-of-equilibrium; Hurwicz, 1986).

equilibrium strategies are straightforward. The players send 2-dimensional messages to report (honestly or not) their marginal willingness to pay and their demand for the public good. In a Nash equilibrium, the outcome of the mechanism is a Lindahl allocation and through their messages, the players reveal directly and honestly their personal prices and (uniform) demand for the public good.

We also deal with the question of emergence of a Nash equilibrium, in case where the players have imperfect information about their economic environment. Following Kim (1993), we use the gradient process to formalize out-of-equilibrium behaviours of the players. We show that the Nash equilibrium is globally stable in quasi-linear economies, under the gradient process.

The remainder of the paper is organized as follows. Section 2 formalizes the economic model. (We consider economies with one private good and one public good, the latter being produced under constant returns to scale from the former.) In Section 3, we define the mechanism and discuss its interpretation. Section 4 deals with the result on implementation of Lindahl allocations. In Section 5, we prove global stability under the gradient process. Section 6 concludes the paper.

2. The model.

We consider an economy with one private good x (the numeraire), one public good y, n consumers, indexed i, and one producer. Each consumer is characterized by a consumption set $X_i = \mathbb{R}^2$, a preference ordering $\boldsymbol{R_i}$, defined over X_i, and an initial endowment $w_i > 0$ of the private good (none in the public good). Preferences are assumed complete, transitive and strictly increasing in the private good. The producer is characterized by a production set Y. We assume that he produces the public good with the private one under constant returns to scale technology. We normalize the units so that one unit of public good costs one unit of private good.

We will denote E the set of such economies, with elements e written as follows:

$$e = (\{1, \ldots, n\}, (X_i, \boldsymbol{R_i}, w_i)_{i=1}^{n}, Y = \{(x, y) \; ; x + y \leq 0\}).$$

Remark. Below, the producer will be supposed to supply any requested amount of the public good, at a price equal to 1. This is justified if we assume that he is representative of a large number of identical producers, supplying the public good on a competitive market.

Allocations. An allocation is a vector $((x_i)_{i=1}^{n}, y) \in \mathbb{R}^{n+1}$, giving the consumptions of the private and public goods by each consumer. For a given economy e, an allocation is said possible if:

$$(x_i, y) \in X_i, \text{ for all } i, \text{ and } \Sigma_{i=1}^{n} x_i + y = \Sigma_{i=1}^{n} w_i.$$

Mechanisms. A mechanism is a pair (M, ρ), consisting of a message space M and an outcome function ρ mapping messages into allocations. The outcome function ρ associates to any message $m \in M$, an allocation $\rho(m) \in \mathbb{R}^{n+1}$ (4).

The pair (M, ρ) defines a game form, where the players are the consumers $i = 1, \ldots, n$, the strategic set of i is M_i (with $\times_{i=1}^{n} M_i := M$), and his preference over M, denoted $\boldsymbol{R_i}$*, follows from his preference $\boldsymbol{R_i}$ over X_i and from the outcome function ρ (5). A strategy of a player i will be denoted m_i.

(4) In fact, $\rho(m)$ also depends on e, through the initial endowments $(w_i)_{i=1}^{n}$. A rigorous notation would thus be $\rho(m, e)$. To simplify, we did not adopt it. Note, however, that $(w_i)_{i=1}^{n}$ do not need to be known.

(5) Formally, for any two messages m et m', the preferences over M are defined by: $m' \, \boldsymbol{R_i}^* \, m \Leftrightarrow \rho(m') \, \boldsymbol{R_i} \, \rho(m)$.

Under perfect information, a natural solution concept for (M, ρ) is the Nash equilibrium (NE), i.e. a joint strategy m^* such that, for all i:

m^* $\boldsymbol{R_i^*}$ (m^*/m_i), for all $m_i \in M_i$,

where: $(m^*/m_i) = (m_1^*, \ldots, m_i, \ldots, m_n^*)$.

Lindahl equilibriums. A Lindahl equilibrium (LE) of e is a vector of personal prices $(p_i^*)_{i=1}^n$, with $\Sigma_{i=1}^n p_i^* = 1$, and a *possible* allocation $((x_i^*)_{i=1}^n, y^*)$, such that, for all i:

(x_i^*, y^*) $\boldsymbol{R_i}$ (x_i, y), for all $(x_i, y) \in X_i$ such that $x_i + p_i^* y \leq w_i$.

The allocation $((x_i^*)_{i=1}^n, y^*)$ is called a Lindahl allocation.

3. The mechanism.

For $n \geq 2$, consider the mechanism (M, ρ) such that:

- the message space is $M = \mathbb{R}^{2n}$;
- the strategic space of player i is $M_i = \mathbb{R}^2$, with elements denoted $m_i = (p_i, y_i)$;
- for all $m \in M$, the outcome function is defined by:

$$\rho(m) = ((w_i - t_i(m))_{i=1}^n, y(m)),$$

where: $$t_i(m) = (1 - \Sigma_{j\neq i} p_j)\, y(m) - (1/2)\, (\Sigma_{i=1}^n p_i - 1 + \Sigma_{j\neq i} y_j/(n-1) - y_i)^2, \quad (i = 1, \ldots, n)$$

$$y(m) = (1/n)\, \Sigma_{i=1}^n y_i.$$

Below, it will be convenient to define, for all m:

$$p_i(m) = 1 - \Sigma_{j\neq i} p_j,$$

$$f_i(m) = \Sigma_{i=1}^n p_i - 1 + \Sigma_{j\neq i} y_j/(n-1) - y_i.$$

With these notations, we have:

$$t_i(m) = p_i(m)\, y(m) - (1/2)\, f_i(m)^2, \qquad (i = 1, 2, \ldots, n).$$

As we will prove below, for all i, the strategies p_i et y_i should be interpreted as player i's report, respectively, of his marginal willingness to pay and his demand for the public good. The players contribute to the cost of provision of the public good, by paying a personal price $p_i(m)$, on each unit, and, possibly, a penalty $(1/2)\, f_i(m)^2$. The personal price $p_i(m)$ of player i is the difference between the marginal cost of production of the public good and the sum of the marginal propensities to pay announced by the other players, i.e. $1 - \Sigma_{j\neq i} p_j$. The supply of public good $y(m)$ is set equal to the average demand, i.e. $(1/n)\, \Sigma_{i=1}^n y_i$.

Our interpretation partly owes its coherence to the properties of the functions $f_i(m)$ $(i = 1, \ldots, n)$.

To see this, consider a strategy m, such that all players avoid the penalty, i.e. $(1/2)\, f_i(m)^2 = 0$, for all i (we show below that this is the case at Nash equilibrium). Then, m solves the linear system:

$$f_i(m) = \Sigma_{i=1}^n p_i - 1 + \Sigma_{j\neq i} y_j/(n-1) - y_i = 0. \qquad (i = 1, 2, \ldots, n) \qquad (1)$$

It is immediate to show that a solution to (1) satisfies:

$$\Sigma_{i=1}^n p_i = 1 \text{ and } y_1 = y_2 = \ldots = y_n = y, \qquad (2)$$

where y is any real number.

Now, if m satisfies (2), it seems natural, for all i, to use the reported marginal propensities to pay p_i to set the personal prices of the players, then to ask them to pay the amount $y_i = y$ they requested, and finally to supply y units of the public good. This is exactly what the outcome function does. Formally, for any m solution to (2), we have:

$$p_i(m) = p_i \text{ et } y(m) = y_i, \text{ for all } i. \qquad (3)$$

Thus, in order to avoid the penalty, the players have to coordinate their messages, in the sense of (2). And if they manage to do so, the outcome function ρ precisely implements what they desire, according to our interpretation of the strategic space (Cf. (3)).

The above interpretation also owes its coherence to the incentives induced by (M, ρ).

To obtain this, let us make two simple statements. Firstly, under the Nash conjecture, player i takes his personal price $p_i(m)$ as given, since it depends on others' strategies only. Secondly, he is always able, by his choice of y_i, to set freely the supply of the public good $y(m)$, and then, by his choice of p_i, to cancel the penalty $(1/2)\, f_i(m)^2$.

From this, at a Nash equilibrium m^*: 1) all players will avoid the penalty, i.e. $f_i(m^*) = 0$, for all i; and 2) $y(m^*)$ will maximise the preferences of all players, given their personal prices and initial endowments; 3) moreover, as $f_i(m^*) = 0$, for all i, implies $\Sigma_{i=1}^{n} p_i(m^*) = 1$ (Cf. (2)), the personal prices $(p_i(m^*))_{i=1}^{n}$ and the allocation $\rho(m^*)$ define a Lindahl equilibrium.

4. Implementation.

We now provide our result on implementation, which was merely sketched in Section 3.

Theorem 1. For all $e \in E$, if $m^* = (p_i^*, y_i^*)_{i=1}^{n}$ is a Nash equilibrium of (M, ρ), then:

(i) $p_i^* = p_i(m^*)$ et $y_i^* = y(m^*)$, for all i;

(ii) $(p_i(m^*))_{i=1}^{n}$ and $\rho(m^*)$ is a Lindahl equilibrium of e.

Proof. Consider an economy $e \in E$ and a NE $m^* = (p_i^*, y_i^*)_{i=1}^{n}$ of (M, ρ).

(i) Assume, by way of contradiction, that there exists i such that $f_i(m^*) \neq 0$. Letting $m_i = (y_i^* - \Sigma_{j\neq i}\, y_j^*/(n-1) - \Sigma_{j\neq i}\, p_j^* + 1, y_i^*)$, we get:

$$p_i(m^*/m_i) = p_i(m^*) = 1 - \Sigma_{j\neq i}\, p_j^*;$$
$$f_i(m^*/m_i)^2 = 0 < f_i(m^*)^2;$$
$$y(m^*/m_i) = y^*(m^*).$$

Hence, by choosing m_i, player i pays less and consumes the same amount of the public good. Since $\boldsymbol{R_i}$ is strictly increasing in the private good, it follows that:

$$(w_i - p_i(m^*/m_i)\, y(m^*/m_i) - (1/2)\, f_i(m^*/m_i)^2, y(m^*/m_i))$$
$$\boldsymbol{P_i}\ (w_i - p_i(m^*)\, y(m^*) - (1/2)\, f_i(m^*)^2, y(m^*)).$$

Therefore, $(m^*/m_i)\ \boldsymbol{P_i^*}\ m^*$ and m^* is not a NE.

By contradiction, we have thus shown that:

$$f_i(m^*) = 0, \text{ for all } i. \qquad (4)$$

Now, notice that $\Sigma_{i=1}^{n} f_i(m) = n\, (\Sigma_{i=1}^{n} p_i - 1)$, for all m. With (4), it follows that:

$$\Sigma_{i=1}^{n} p_i^* = 1. \qquad (5)$$

Substituting in each $f_i(m^*)$, we get:

$$f_i(m^*) = \Sigma_{j\neq i}\, y_j^* - (n-1)\, y_i^* = 0, \text{ for all } i. \qquad (6)$$

It follows directly from (4), (5) and (6) that, for all i:

$$p_i^* = 1 - \Sigma_{j\neq i}\, p_j^* = p_i(m^*); \qquad (7)$$
$$y_i^* = (1/n)\, \Sigma_{i=1}^{n} y_i^* = y^*(m^*). \qquad (8)$$

(ii) Let $((x_i^*)_{i=1}^{n}, y^*) = \rho(m^*)$. Since $f_i(m^*) = 0$, for all i (Cf. (4)), we have:

$$((x_i^*)_{i=1}^{n}, y^*) = ((w_i - p_i(m^*)\, y(m^*))_{i=1}^{n}, y(m^*)).$$

This allocation is possible, for $\Sigma_{i=1}^{n} p_i(m^*) = \Sigma_{i=1}^{n} p_i^* = 1$ (Cf. (5) et (7)) implies that:

$$\Sigma_{i=1}^{n} x_i^* + y^* = \Sigma_{i=1}^{n} (w_i - p_i(m^*)\, y^*) + y^* = \Sigma_{i=1}^{n} w_i + (1 - \Sigma_{i=1}^{n} p_i(m^*))\, y^* = \Sigma_{i=1}^{n} w_i.$$

Since the players are never satiated in the private good, without loss of generality, assume (by contradiction) that i and (x_i, y) exist such that:

$$(x_i, y)\ \boldsymbol{P_i}\ (x_i^*, y^*) \text{ and } x_i + p_i(m^*)\, y = w_i.$$

Substituting (using $x_i = w_i - p_i(m^*)\, y$ and $(x_i^*, y^*) = (w_i - p_i(m^*)\, y(m^*), y(m^*))$), we find:

$$(w_i - p_i(m^*)\, y, y)\ \boldsymbol{P_i}\ (w_i - p_i(m^*)\, y(m^*), y(m^*)).$$

Letting $m_i = (n\,(y - \Sigma_{j\neq i}\, y_j^*/(n-1)) - \Sigma_{j\neq i}\, p_j^* + 1,\ n\, y - \Sigma_{j\neq i}\, y_j^*)$, we get:

$$p_i(m^*/m_i) = p_i(m^*) = 1 - \Sigma_{j\neq i}\, p_j^*;$$
$$f_i(m^*/m_i)^2 = f_i(m^*)^2 = 0;$$
$$y(m^*/m_i) = y.$$

Substituting, we obtain:

$$(w_i - p_i(m^*/m_i)\, y(m^*/m_i) - (1/2)\, f_i(m^*/m_i)^2, y(m^*/m_i))$$
$$\boldsymbol{P_i}\ (w_i - p_i(m^*)\, y(m^*) - (1/2)\, f_i(m^*)^2, y(m^*)).$$

Therefore, $(m^*/m_i)\ \boldsymbol{P_i^*}\ m^*$ and m^* is not a NE.

Finally, $((x_i^*)_{i=1}^{n}, y^*)$ is possible and satisfy, for all i: $(x_i^*, y^*)\ \boldsymbol{R_i}\ (x_i, y)$, for all (x_i, y) such that $x_i + p_i(m^*)\, y \leq w_i$. QED

Theorem 2. For all $e \in E$, if $(p_i^*)_{i=1}^{n}$ and $((x_i^*)_{i=1}^{n}, y^*)$ is a Lindahl equilibrium of e, then $m^* = (p_i^*, y^*)_{i=1}^{n}$ is the unique Nash equilibrium of (M, ρ) such that $\rho(m^*) = ((x_i^*)_{i=1}^{n}, y^*)$.

Proof. Consider an economy $e \in E$ and a LE $(p_i^*)_{i=1}^{n}$ et $((x_i^*)_{i=1}^{n}, y^*)$ of e. By definition:

$$\Sigma_{i=1}^{n}\, p_i^* = 1;$$
$$\Sigma_{i=1}^{n}\, x_i^* + y^* = \Sigma_{i=1}^{n}\, w_i;$$
$$(x_i^*, y^*)\ \boldsymbol{R_i}\ (x_i, y) \text{ for all } (x_i, y) \in X_i \text{ such that } x_i + p_i^*\, y \leq w_i, \text{ for all } i. \quad (9)$$

Because the players are never satiated with the private good, (9) implies, for all i and all y:

$$(w_i - p_i^*\, y^*, y^*)\ \boldsymbol{R_i}\ (w_i - p_i^*\, y, y). \quad (10)$$

Letting $m^* = (p_i^*, y^*)_{i=1}^{n}$, we obtain:

$$p_i(m^*) = 1 - \Sigma_{j\neq i}\, p_j^* = p_i^* \text{ (for } \Sigma_{i=1}^{n}\, p_i^* = 1);$$
$$f_i(m^*)^2 = 0 \text{ (for } \Sigma_{i=1}^{n}\, p_i^* = 1 \text{ and } y_i^* = \Sigma_{j\neq i}\, y_j^*/(n-1));$$
$$y(m^*) = y^*.$$

Substituting in (10), we get, for all i and all y:

$$(w_i - p_i(m^*)\, y(m^*) - (1/2)\, f_i(m^*)^2, y(m^*))\ \boldsymbol{R_i}\ (w_i - p_i(m^*)\, y, y). \quad (11)$$

For all i, let now $m_i = (p_i, n\, y - \Sigma_{j\neq i}\, y^*)$, where p_i is any real number. By definition:

$$p_i(m^*/m_i) = p_i(m^*) = 1 - \Sigma_{j\neq i}\, p_j^*;$$
$$f_i(m^*/m_i)^2 \geq f_i(m^*)^2 = 0;$$
$$y^*(m^*/m_i) = y.$$

Substitute $p_i(m^*/m_i) = p_i(m^*)$ and $y^*(m^*/m_i) = y$ into the RHS of (11). Then, for all i and m_i:

$$(w_i - p_i(m^*)\, y(m^*) - (1/2)\, f_i(m^*)^2, y(m^*))\ \boldsymbol{R_i}\ (w_i - p_i(m^*/m_i)\, y(m^*/m_i), y(m^*/m_i)). \quad (12)$$

As $\boldsymbol{R_i}$ is strictly increasing in the private good and $f_i(m^*/m_i)^2 \geq 0$, we have, for all i and all m_i:

$$(w_i - p_i(m^*/m_i)\, y(m^*/m_i), y(m^*/m_i))$$
$$\boldsymbol{R_i}\ (w_i - p_i(m^*/m_i)\, y(m^*/m_i) - (1/2)\, f_i(m^*/m_i)^2, y(m^*/m_i)). \quad (13)$$

By transitivity of $\boldsymbol{R_i}$:

$$(w_i - p_i(m^*)\, y(m^*) - (1/2)\, f_i(m^*)^2, y(m^*))$$
$$\boldsymbol{R_i}\ (w_i - p_i(m^*/m_i)\, y(m^*/m_i) - (1/2)\, f_i(m^*/m_i)^2, y(m^*/m_i)). \quad (14)$$

Therefore, for all i:

$$m^*\ \boldsymbol{R_i^*}\ (m^*/m_i), \text{ for all } m_i \in M_i,$$

which means that m^* is a NE of (M, ρ). QED

Thus, (M, ρ) belongs to the class of game forms which fully implement with Nash equilibrium the Lindahl social choice rule ([6]). Other mechanisms from this class are Hurwicz (1979), Kim (1993), Tian (1989), de Trenqualye (1994) and Walker (1981).

However, none has the further property of direct revelation of the underlying Lindahl equilibrium, which is thus characteristic of (M, ρ). Indeed, as Theorems 1 and 2 attempt to make as clear as possible, at a Nash equilibrium, not only (M, ρ) yields a Lindahl allocation, but it also induces the players to report truthfully their underlying personal prices and (uniform) demand for the public good.

Remark. A player's Nash equilibrium strategy, besides his personal characteristics, relies on aggregated characteristics of the other players only. This property is possible because (M, ρ) is anonymous (i.e., any permutation of the players leaves the outcome unchanged). Among the mechanisms above, only Kim (1993) is anonymous.

Remark. The assumption $X_i = \mathbb{R}^2$, for all i, is necessary for the game form (M, ρ) to be well-defined. For example, if $X_i = \mathbb{R}^2_+$, there exists some strategic profiles m leaving i outside his consumption set X_i. Among the mechanisms above, only Tian (1989) avoids this weakness ([7]).

Remark. Our mechanism fails to be globally (or weakly) feasible, i.e. $\Sigma_{i=1}^{n} t_i(m) \geq y(m)$ is not satisfied for some m ([8]). However, a mechanism based on the same principles as (M, ρ) can be built, by letting:

$$t_i(m) = p_i(m)\, y(m) + (1 - 1/n)\, |f_i(m)\, y(m)|, \text{ for all } i,$$

which is globally (or weakly) feasible and still satisfies Theorems 1 and 2. Feasibility is easy to prove, noticing that $\Sigma_{i=1}^{n} |f_i(m)\, y(m)| \geq \Sigma_{i=1}^{n} f_i(m)\, y(m) = n\, (\Sigma_{i=1}^{n} p_i - 1)\, y(m)$, for all m. The proofs above can be adapted to show Theorems 1 and 2, with only negligible changes. (The only difference is that the case where $y(m) = 0$ at a Nash equilibrium must be considered in isolation.) The reason why we did not pay much attention to this version of our mechanism is our focus on the stability result in the next section.

5. Global Stability.

In most situations of economic interest, the players actually have imperfect information about their economic environment. In such case, the Nash equilibrium remains relevant as a solution concept, provided that it arises as a stationary point of some decentralized adjustment process. Following Kim (1993), we formalize this idea using the gradient process. We show that, in quasi-linear economies, the associated dynamic is globally stable and the (necessarily unique) stationary point is the (unique) Nash equilibrium of (M, ρ).

Let us consider an economy e, where the preferences $\boldsymbol{R_i}$ of all consumers can be represented by differentiable utility functions $U_i(x_i, y)$ ($i = 1, \ldots, n$). Under (M, ρ), for any strategic profile m in $\mathbb{R}^{2n}$, consumer i's utility is given by:

$$u_i(m) := U_i(w_i - p_i(m)\, y(m) - (1/2)\, f_i(m)^2, y(m)),$$

and is differentiable.

([6]) A mechanism is said to fully implement the Lindahl social choice rule if, for all $e \in E$, the set of allocations implementable as Nash equilibrium and the set of Lindahl allocations are equal (de Trenqualye, 1994).

([7]) Hurwicz (1979) obtains a well-defined game by constructing an extended preference ordering for each player, such that any $(x_i, y) \in X_i$ is superior to all $(x_i, y) \notin X_i$.

([8]) Since (M, ρ) deals with the case of economies with two agents (or more) and is smooth, we know by theorem that it cannot be balanced (Hurwicz, 1986).

The gradient process describes the behaviours of players who adjust their strategy $m_i = (p_i, y_i)$, in the direction that maximises the instantaneous increase of their utility $u_i(m)$, taking others' strategies as given. Formally, let us denote:

- $\mathrm{d}m_i/\mathrm{d}t = (\mathrm{d}p_i/\mathrm{d}t, \mathrm{d}y_i/\mathrm{d}t)$, the instantaneous variation of i's strategy;
- $F_i(m; e) = (\partial u_i(m)/\partial p_i, \partial u_i(m)/\partial y_i)$, the gradient of i's utility, with respect to p_i and y_i;

The gradient process is formalized as the system of differential equations:

$$\mathrm{d}m/\mathrm{d}t := (\mathrm{d}m_i/\mathrm{d}t)_{i=1}^{n} = (F_i(m; e))_{i=1}^{n} := F(m; e), \qquad m(0) = m_0, \qquad (S)$$

where $m_0 = (p_{0i}, y_{0i})_{i=1}^{n}$ is the initial condition.

Given a strategic profile m_0 reported at time $t = 0$, the solution of (S) such that $m(0) = m_0$, if it exists, gives the evolution of the economy. A point m^* in IR^{2n} is called a stationary point of (S), for an economy e, if $F(m^*; e) = 0$. It is said globally stable if, whatever the initial condition m_0 may be, the solution to (S), such that $m(0) = m_0$, exists and tends to m^* as t tends to infinity. (If a stationary point is globally stable, it is necessarily unique).

Theorem 3. Let e be a quasi-linear economy, i.e. an economy where, for all i:

$$U_i(x_i, y) = x_i + v_i(y), \text{ with } v_i'(y) > 0 > v_i''(y), \text{ for all } y \geq 0.$$

If one exists, the economy e has a unique Lindahl equilibrium $(p_i^*)_{i=1}^{n}$ and $((x_i^*)_{i=1}^{n}, y^*)$. The (unique) stationary point of (S) is globally stable. It coincides with the (unique) Nash equilibrium $m^* = (p_i^*, y^*)_{i=1}^{n}$ of (M, ρ), such that $\rho(m^*) = ((x_i^*)_{i=1}^{n}, y^*)$.

Proof. Consider a quasi-linear economy e. Assume that $(p_i^*)_{i=1}^{n}$ and $((x_i^*)_{i=1}^{n}, y^*)$ is a LE for e. By definition:

$$p_i^* = v_i'(y^*) \text{ et } x_i^* + p_i^* y^* = w_i, \text{ for all } i,$$
$$\Sigma_{i=1}^{n} p_i^* = \Sigma_{i=1}^{n} v_i'(y^*) = 1.$$

Since the functions $v_i'(y)$ $(i = 1, \ldots, n)$ are decreasing, this system has at most one solution.

Under (M, ρ), for all m, player i's utility is equal to:

$$u_i(m) = w_i - (1 - \Sigma_{j\neq i} p_j)\, \Sigma_{i=1}^{n} y_i/n$$
$$- (1/2)\, (\Sigma_{i=1}^{n} p_i - 1 + \Sigma_{j\neq i} y_j/(n-1) - y_i)^2 + v_i(\Sigma_{i=1}^{n} y_i/n).$$

Hence, the dynamical system (S) is given, for all i, by:

$$\mathrm{d}p_i/\mathrm{d}t = -(\Sigma_{i=1}^{n} p_i - 1 + \Sigma_{j\neq i} y_j/(n-1) - y_i),$$
$$\mathrm{d}y_i/\mathrm{d}t = (1/n)\, [v_i'(\Sigma_{i=1}^{n} y_i/n) - (1 - \Sigma_{j\neq i} p_j)] + (\Sigma_{i=1}^{n} p_i - 1 + \Sigma_{j\neq i} y_j/(n-1) - y_i).$$

By construction, a stationary point m^* of (S) satisfies the (first-order) necessary conditions for the maximization of $u_i(m)$, with respect to p_i and y_i, for all i. In a quasi-linear economy, these conditions are also sufficient. Hence, a stationary point of (S) is a NE of (M, ρ). By Theorem 1, a NE is associated to a LE of e, thus to $(p_i^*)_{i=1}^{n}$ and $((x_i^*)_{i=1}^{n}, y^*)$ (by uniqueness). By Theorem 2, $(p_i^*, y^*)_{i=1}^{n}$ is the unique NE implementing it. Therefore, $m^* = (p_i^*, y^*)_{i=1}^{n}$ is the unique stationary point of (S).

Define $(q_i, z_i)_{i=1}^{n}$ as:

$$q_1 = \Sigma_{i=1}^{n} p_i - 1, \qquad z_1 = \Sigma_{i=1}^{n} y_i/n - y^*, \qquad (i = 1)$$
$$q_i = (p_i - p_i^*) + n\,(y_i - y^*), \qquad z_i = (n-1)\,(p_i - p_i^*) + n\,(y_i - y^*). \qquad (i = 2, \ldots, n)$$

We can now restate (S) in terms of $(q_i, z_i)_{i=1}^{n}$:

$$\mathrm{d}q_1/\mathrm{d}t = -n\, q_1,$$
$$\mathrm{d}z_1/\mathrm{d}t = (1/n^2)\, [\alpha(n)\, q_1 + \Sigma_{i=1}^{n} v_i'(y^* + z_1) - 1], \qquad (S_1)$$
$$\mathrm{d}q_i/\mathrm{d}t = n\, q_1 + n\, z_1 - q_i + v_i'(y^* + z_1) - p_i^*,$$

$dz_i/dt = 2\, q_1 + n\, z_1 - z_i/(n-1) + v_i'(y^* + z_1) - p_i^*$. $(S_i;\ i = 2, \ldots, n)$
where: $\alpha(n) = (n^2 + n - 1)$.

The subsystem (S_1) is independent of $(q_i, z_i)_{i=2}^n$. And each subsystem (S_i) ($i = 2, \ldots, n$) depends on q_1, z_1, q_i and z_i, only. The proof below takes advantage of this recursive structure.

Define the function:

$$w(q_1, z_1) = -(n/2)\,(q_1)^2 + \Sigma_{i=1}^n v_i(y^* + z_1) - (y^* + z_1).$$

It has a global maximum at $(q_1, z_1) = (0, 0)$. Note that this point is the unique stationary point of (S_1). If (q_1, z_1) satisfies the dynamical system (S_1), then:

$$(dw/dt)\,(q_1, z_1) = -n\, q_1\, dq_1/dt + (\Sigma_{i=1}^n v_i'(y^* + z_1) - 1)\, dz_1/dt,$$
$$= (1/n^2)\,[n^4\,(q_1)^2 + (\Sigma_{i=1}^n v_i'(y^* + z_1) - 1)(\alpha(n)\, q_1 + \Sigma_{i=1}^n v_i'(y^* + z_1) - 1)],$$
$$= (1/n^2)\,\{(n^4 - (\alpha(n)/2)^2)\,(q_1)^2 + [(\alpha(n)/2)\, q_1 + \Sigma_{i=1}^n v_i'(y^* + z_1) - 1]^2\}.$$

Notice that $(dw/dt)\,(0, 0) = 0$ and $(dw/dt)\,(q_1, z_1) > 0$, for all $(q_1, z_1) \neq (0, 0)$ (for $n^4 - (\alpha(n)/2)^2 > 0$). Hence, $w(q_1, z_1)$ is a Lyapunov function for the dynamical system (S_1). By theorem, the stationary point $(q_1, z_1) = (0, 0)$ of (S_1) is globally stable.

Now consider the subsystems (S_i) ($i = 2, \ldots, n$). When (q_1, z_1) satisfies (S_1), they are composed of two differential equations, each having the generic form:

$$dX(t)/dt = -a\, X(t) + b(t), \qquad X(0) = X_0. \tag{15}$$

where $a > 0$ and $b(t)$ is continuous over $\mathbb{R}^+$ and $\to 0$ as $t \to \infty$ (since $(q_1, z_1) \to (0, 0)$ as $t \to \infty$). It is clear that if $X(t)$ is solution to (15), then $X(t) \to 0$ as $t \to \infty$ ([9]).

Finally, we have shown that $(q_i, z_i)_{i=1}^n \to (0, 0)_{i=1}^n$ as $t \to \infty$. Therefore, $m \to (p_i^*, y^*)_{i=1}^n$ as $t \to \infty$. QED

Remark. Immediate corollaries follow from the proof of Theorem 3. On the one hand, with initial conditions m_0 satisfying $\Sigma_{i=1}^n p_{0i} = 1$, the outcomes will remain balanced during the whole path of convergence to the stationary point m^*. Indeed, (S_1) implies that $q_1 = \Sigma_{i=1}^n p_i - 1 = 0$, for all t. On the other hand, the sum of the players' utility will increase continuously with time if the initial conditions and the path of convergence stay close to m^*. Indeed, in a neighbourhood of m^*, $d[\Sigma_{i=1}^n u_i(m)]/dt$ is approximately equal to $dw(q_1, z_1)/dt$ and we have shown that $dw(q_1, z_1)/dt > 0$, for all t.

6. Conclusion.

We have shown that Lindahl allocations can be implemented with a mechanism such that in a Nash equilibrium, each players' equilibrium strategy is simply to reveal his personal price and his demand for the public good (in Theorems 1 and 2). This property means the existence of one straightforward way to interpret and process the messages used in the mechanism. We see this as valuable, for intuitively this kind of game forms could also suit players with bounded rationality. We have also shown that in quasi-linear economies, the Nash equilibrium is globally stable under the 'gradient' process (in Theorem 3). This implies that players with imperfect information about their environment could nevertheless implement a Lindahl

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([9]) Indeed, the solution to (15) is: $X(t) = e^{-at}\,[X_0 + \int_0^t e^{as}\, b(s)\, ds]$. We have: $|X(t)| \leq e^{-at}\,[|X_0| + \int_0^t e^{as}\, |b(s)|\, ds]$. Since $b(t) \to 0$ as $t \to \infty$, for all $\varepsilon_0 = a\varepsilon/2 > 0$, there exists t_0 such that, for all $t > t_0$, $|b(t)| < \varepsilon_0$. Let $A = \mathrm{Sup}\{|b(t)|\,;\ t \geq 0\}$ ($A < \infty$, for $b(t)$ is continuous and tends to 0). For all $t > t_0$, we show:

$$|X(t)| < e^{-at}\,[|X_0| + A \int_0^{t_0} e^{as}\, ds + \varepsilon_0 \int_{t_0}^t e^{as}\, ds] = e^{-at}\,\{|X_0| + (1/a)\,[A\,(e^{at_0} - 1) + \varepsilon_0\,(e^{at} - e^{at_0})]\}.$$

The RHS tends to $\varepsilon_0/a = \varepsilon/2$ when t tends to infinity. Hence, $|X(t)|$ will be smaller than ε for t large enough. Since ε can be chosen as small as desired, $X(t) \to 0$ as $t \to \infty$.

allocation using the game form. A future extension of this paper could be to test these intuitions in an experimental setting. In particular, interesting results may be obtained in a situation where two groups play the game, but only one is explicitly taught about the way to interpret the messages.

7. References.

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